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On the existence of parallel flow for mixed convection in an inclined duct

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Abstract

The necessary condition for the occurrence of parallel mixed convection flow in an inclined duct is determined by employing the Boussinesq approximation. A sample case involving an inclined infinitely-wide plane channel is discussed to illustrate this condition. It is shown that, according to the necessary condition, parallel flow cannot occur in this case. Indeed, the investigated flow is the superposition of a parallel streamwise flow and a secondary flow. An exponential equation of state for the fluid is assumed and the balance equations are solved analytically to determine the dimensionless velocity distribution, as well as the conditions for the occurrence of flow reversal.

Keywords: Laminar flow; Mixed convection; Boussinesq approximation; Inclined duct; Analytical methods

1. Introduction

Several theoretical and experimental investigations of laminar buoyancy-induced flows in vertical or inclined ducts are available in the literature. Earlier theoretical papers [1–3] are based on analytical solutions of the balance equations and point out the basic features of laminar mixed-convection flows in the fully developed regime of vertical ducts. These papers refer to the simplest cross-sectional shapes, i.e. plane-parallel channels, circular tubes and rectangular ducts. In Ref. [4], an interesting extension of the solution found in Ref. [1] for a plane-parallel vertical channel with isothermal walls having unequal temperatures is obtained. The authors release the Boussinesq approximation invoked in Ref. [1] and assume that the fluid properties change with temperature according to an ideal gas model.

In the last fifteen years, the analysis of mixed convection flows in vertical and inclined ducts has been the subject of several papers, mainly following the interest of these flows for engineering problems such as the cooling of electronic equipments and the design of solar collectors. The investigations presented in Refs. [5–9] are devoted to the analysis of either developing or fully developed flows, and cases such that flow reversal occurs are considered. In Ref. [7], an analytical solution based on Fourier series expansions is presented which yields the velocity and temperature field for fully-developed mixed convection in a vertical rectangular duct with a hotter isothermal wall and three cooler isothermal walls. In Ref. [8], a numerical solution of the balance equations is obtained for buoyancy-induced heat and mass transfer

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Nomenclature

A, B	quantities defined in Eq. (13)
b	distance between the channel walls
c_v	specific heat at constant volume
D	$=2b$, hydraulic diameter
$F(Y)$	dimensionless function defined by Eq. (32)
$\mathbf{g}$	gravitational acceleration
Gr	Grashof number, defined in Eq. (31)
k	thermal conductivity
p	pressure
P	difference between the pressure and the hydrostatic pressure
$\mathbf{r}$	position vector
Ra	Rayleigh number
Re	Reynolds number, defined in Eq. (31)
T	temperature
T_0	reference temperature
u, v	dimensionless velocity components defined in Eq. (31)
$\mathbf{U}$	fluid velocity
U_0	mean fluid velocity in a channel section
x, y, z	rectangular coordinates
Y	dimensionless coordinate defined in Eq. (31)

Greek symbols

β	volumetric coefficient of thermal expansion
Γ	ratio between Gr and Re
Γ_{fr}, $\widetilde{\Gamma}_{fr}$	threshold values of Γ for the onset of flow reversal, given by Eqs. (41) and (42)
$\Delta\varrho(T)$	difference between the mass density and the reference mass density
Λ	dimensionless parameter defined by Eq. (31)
μ	dynamic viscosity
Ξ	dimensionless parameter defined by Eq. (31)
ϱ	mass density
ϱ_0	reference mass density, i.e. mass density for $T = T_0$
φ	tilt angle defined by Eq. (21)
Φ	viscous dissipation function defined by Eq. (4)
$\psi(Y)$	local bending angle defined by Eq. (43)

Superscript

$'$	projection of a vector on the xy-plane

in a vertical rectangular duct such that three walls are adiabatic, while the fourth is kept at a uniform temperature or at a uniform heat flux. In Ref. [9], the laminar and parallel buoyancy-induced flow in a vertical rectangular duct is considered, providing also a theorem on the uniqueness of the parallel flow solution in vertical ducts of arbitrary shape.

The effect of viscous dissipation for parallel mixed convection flows is analysed either in a vertical plane channel [10] or in a vertical circular duct [11]; both solutions are obtained utilising a perturbation method. A perturbation series solution [12] refers to the case of combined forced and free flow with viscous dissipation in an inclined plane channel with isothermal walls having unequal temperatures. In this paper, it is shown that the tilt angle, the viscous dissipation effect and the buoyancy effect influence the distribution of the difference between the pressure and the hydrostatic pressure in a channel cross-section: this distribution is uniform for a vertical channel while it becomes nonuniform when the channel is inclined.

The main aim of the present paper is to state and prove a theorem defining the necessary condition for the occurrence of fully-developed parallel flow in an inclined duct with an arbitrary cross section. This theorem holds under the assumption of validity of the Boussinesq approximation as well as under the hypothesis that the thermal boundary conditions do not produce a net fluid heating in the axial direction. In order to illustrate the importance of this theorem, an example is discussed. The example refers to an inclined plane channel not fulfilling the necessary condition for the occurrence of parallel flow. Indeed, the velocity field is helicoidal, i.e. a secondary flow occurs. An analytical solution is obtained, without invoking a linear equation of state, but assuming a more general exponential relation between density and temperature. This equation of state reduces to the usual linear relation when very small temperature differences are present within the fluid.

2. The necessary condition for the existence of parallel flows

Let us consider an inclined duct whose cross section has an arbitrary shape. Moreover, let us choose Cartesian coordinates (x, y, z) such that the z-axis is parallel to the duct axis, while the duct cross section lies on the plane (x, y). In particular, the duct cross section corresponds to a region $\mathfrak{D}$ with boundary $\partial\mathfrak{D}$ on the plane (x, y). According to the Boussinesq approximation, for a stationary flow of a Newtonian fluid, the mass balance equation and the momentum balance equation can be expressed as

$$\nabla \cdot \mathbf{U} = 0, \tag{1}$$

$$\varrho_0 \mathbf{U} \cdot \nabla \mathbf{U} = \varrho(T)\mathbf{g} - \nabla p + \mu \nabla^2 \mathbf{U}. \tag{2}$$

In Eq. (2), $\varrho(T)$ is the temperature-dependent mass density evaluated through the equation of state, while ϱ_0 and μ are the mass density and the dynamic viscosity evaluated at a reference temperature T_0. Finally, the energy balance equation is given by

$$\varrho_0 c_v \mathbf{U}\cdot\nabla T = k\nabla^2 T + \mu\Phi, \tag{3}$$

where Φ is the viscous dissipation function given by

$$\Phi = 2\left[\left(\frac{\partial U_x}{\partial x}\right)^2 + \left(\frac{\partial U_y}{\partial y}\right)^2 + \left(\frac{\partial U_z}{\partial z}\right)^2\right] + \left(\frac{\partial U_y}{\partial x}+\frac{\partial U_x}{\partial y}\right)^2 + \left(\frac{\partial U_z}{\partial y}+\frac{\partial U_y}{\partial z}\right)^2 + \left(\frac{\partial U_x}{\partial z}+\frac{\partial U_z}{\partial x}\right)^2. \tag{4}$$

Let us assume that the thermal boundary conditions are such that:

a) a fully-developed regime where

$$\frac{\partial \mathbf{U}}{\partial z} = 0 \tag{5}$$

exists;

b) no net fluid heating occurs in the fully-developed regime, i.e.

$$\frac{\partial T}{\partial z} = 0. \tag{6}$$

Then, in the fully-developed regime, Eqs. (1)–(3) yield

$$\nabla'\cdot\mathbf{U}' = 0, \tag{7}$$

$$\varrho_0\mathbf{U}'\cdot\nabla'\mathbf{U}' = \Delta\varrho(T)\mathbf{g}' - \nabla' P + \mu\nabla'^2\mathbf{U}', \tag{8}$$

$$\varrho_0\mathbf{U}'\cdot\nabla' U_z = \Delta\varrho(T)g_z - \frac{\partial P}{\partial z} + \mu\nabla'^2 U_z, \tag{9}$$

$$\varrho_0 c_v\mathbf{U}'\cdot\nabla' T = k\nabla'^2 T + \mu\Phi. \tag{10}$$

In Eqs. (7)–(10), the primed vectors are the two-dimensional vectors obtained by projection on the (x,y)-plane, while ∇' is the two-dimensional gradient $(\partial/\partial x, \partial/\partial y)$. Moreover, $\Delta\varrho(T) = \varrho(T) - \varrho_0$ and $P = p - \varrho_0\mathbf{g}\cdot\mathbf{r}$, where $\mathbf{r}$ is the position vector.

If Eqs. (8) and (9) are differentiated with respect to z, one obtains

$$\nabla'\frac{\partial P}{\partial z} = 0, \tag{11}$$

$$\frac{\partial^2 P}{\partial z^2} = 0. \tag{12}$$

Eqs. (11) and (12) imply that there exist a function $A(x,y)$ and a constant B such that

$$P(x,y,z) = A(x,y) + Bz. \tag{13}$$

By substituting Eq. (13) into Eqs. (8) and (9), one obtains

$$\varrho_0\mathbf{U}'\cdot\nabla'\mathbf{U}' = \Delta\varrho(T)\mathbf{g}' - \nabla' A + \mu\nabla'^2\mathbf{U}', \tag{14}$$

$$\varrho_0\mathbf{U}'\cdot\nabla' U_z = \Delta\varrho(T)g_z - B + \mu\nabla'^2 U_z. \tag{15}$$

The following statement holds.

- *A parallel flow solution, i.e. a solution with* $\mathbf{U}' = 0$*, exists in the fully-developed regime only if the temperature field is such that* $\mathbf{g}'\times\nabla' T = 0$*, at every position in the region* $\mathfrak{D}$.

The proof is as follows. If one assumes $\mathbf{U}' = 0$, Eq. (14) yields

$$\Delta\varrho(T)\mathbf{g}' - \nabla' A = 0. \tag{16}$$

Since $\mathbf{g}'$ is a constant vector, by evaluating the two-dimensional curl of both sides of Eq. (16), one obtains

$$0 = \nabla'\times[\Delta\varrho(T)\mathbf{g}' - \nabla' A] = -\mathbf{g}'\times\nabla'\Delta\varrho(T) = -\mathbf{g}'\times\nabla'\varrho(T) = \beta\varrho(T)\mathbf{g}'\times\nabla' T, \tag{17}$$

where β is the coefficient of thermal expansion defined as

$$\beta = -\frac{1}{\varrho}\frac{d\varrho}{dT}. \tag{18}$$

Eq. (17) ensures the validity of the statement.

The above statement implies that, for a non-vertical duct ($\mathbf{g}' \neq 0$), a parallel flow solution for the fully-developed regime can be found only if the thermal boundary conditions are such that either the vector field $\nabla' T$ is a parallel field with the same direction as the vector $\mathbf{g}'$ or the fluid is isothermal, i.e. $\nabla' T = 0$. On the other hand, for a vertical duct ($\mathbf{g}' = 0$), a parallel flow solution always exists.

If, for a given flow, the viscous dissipation term $\mu\Phi$ can be neglected in the energy balance equation, then a parallel flow solution ($\mathbf{U}' = 0$) is such that the temperature field $T(x,y)$ fulfils the Laplace equation, i.e.

$$\nabla'^2 T = 0, \tag{19}$$

as it can be easily inferred from Eq. (10). Therefore, in this case, the thermal boundary conditions determine directly the temperature field among the solutions of Eq. (19) without any interplay with the other balance equations.

3. Physical meaning of the necessary condition for parallel flows

Beyond the mathematical character of the necessary condition proved in Section 2, one could question on

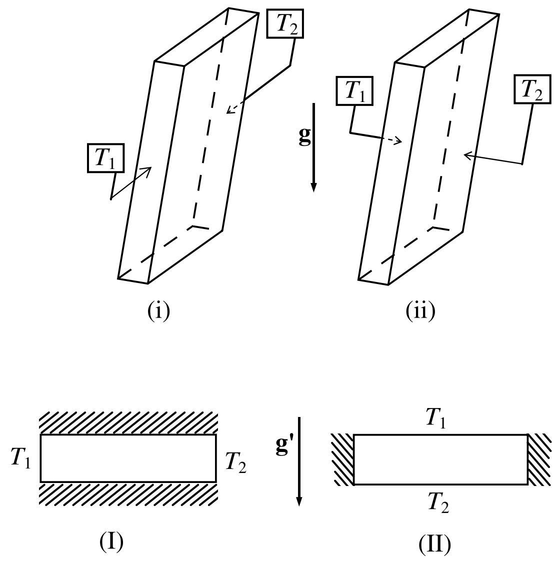

Fig. 1. Drawing of ducts (i) and (ii) and of the corresponding rectangular cavities (I) and (II).

its physical content. A physical approach to this subject can be performed by comparing two different inclined rectangular ducts, duct (i) and duct (ii). A representation of these ducts is given in Fig. 1. Duct (i) is a rectangular duct with two facing adiabatic walls inclined with respect to the gravitational acceleration **g** and two facing isothermal walls with different temperatures parallel to **g**. Duct (ii) is a rectangular duct with two facing adiabatic walls parallel to **g** and two facing isothermal walls with different temperatures inclined with respect to **g**. For both ducts, it is quite obvious that, if one neglects the viscous dissipation effect, one can first determine the secondary-flow velocity field $\mathbf{U}'$ and the temperature field T by solving Eqs. (7), (10) and (14) and then one can evaluate the velocity component U_z by solving Eq. (15). In other words, $\mathbf{U}'$ and T are determined by the set of differential equations

$$\begin{aligned} &\boldsymbol{\nabla}' \cdot \mathbf{U}' = 0, \\ &\varrho_0 c_v \mathbf{U}' \cdot \boldsymbol{\nabla}' T = k \nabla'^2 T, \\ &\varrho_0 \mathbf{U}' \cdot \boldsymbol{\nabla}' \mathbf{U}' = \Delta\varrho(T)\mathbf{g}' - \boldsymbol{\nabla}' A + \mu \nabla'^2 \mathbf{U}'. \end{aligned} \tag{20}$$

Eq. (20) defines a two-dimensional buoyancy-driven flow in a rectangular cavity with an effective gravitational acceleration g'. If duct (i) is considered, the corresponding rectangular cavity, denoted as (I), is such that the isothermal sides are parallel to $\mathbf{g}'$. If duct (ii) is considered, the corresponding rectangular cavity, denoted as (II), is such that the isothermal sides are orthogonal to $\mathbf{g}'$. A drawing of the rectangular cavities (I) and (II) is reported in Fig. 1. Obviously, if ducts (i) and (ii) are vertical, $\mathbf{g}'$ vanishes and no buoyancy-driven flow takes place in the corresponding rectangular cavities, i.e. the secondary-flow velocity $\mathbf{U}'$ is zero. In this case, parallel flow can occur in the vertical ducts (i) and (ii). On the other hand, if ducts (i) and (ii) are inclined so that $\mathbf{g}' \neq 0$, the analysis of laminar flow in the rectangular cavity (I) and in the rectangular cavity (II) is less trivial. In fact, as it has been widely discussed in the literature [13], there is a fundamental difference between the rectangular cavity (I) and the rectangular cavity (II). In the cavity (I), the fluid can never be steadily at rest even for small values of the temperature difference between the isothermal sides, i.e. a nontrivial secondary-flow velocity distribution $\mathbf{U}'$ always exists. Therefore, one concludes that parallel flow cannot exist in duct (i). Indeed, for duct (i), it is quite reasonable that there exists a narrow neighbourhood of each isothermal wall where $\boldsymbol{\nabla}'T$ is orthogonal to the isothermal wall itself, i.e. where $\mathbf{g}' \times \boldsymbol{\nabla}'T \neq 0$. According to the necessary condition proved in Section 2, this implies that parallel flow cannot occur in duct (i). In the cavity (II), a steady velocity distribution such that no fluid motion takes place may exist ($\mathbf{U}' = 0$). The occurrence of this steady distribution does not undergo specific restrictions if the cold isothermal side is below. On the other hand, if the hot isothermal side is below, the steady velocity distribution such that $\mathbf{U}' = 0$ is stable only if the value of the Rayleigh number, $Ra = \varrho_0^2 c_v |\mathbf{g}'| Xi b^3/(\mu k)$ where b is the distance between the isothermal sides, is smaller than a critical value Ra_c which depends on the aspect ratio of the rectangular cavity [13]. To summarize, even if the necessary condition proved in Section 2 predicts the existence of a steady parallel flow solution for duct (ii), this solution can become unstable if the temperature difference between the isothermal walls is not sufficiently small. Indeed, the statement proved in Section 2 provides only a necessary condition for parallel flow, while the occurrence of parallel flow may be ruled out in practice by instabilities.

In the following section, an example of fully-developed mixed convection is analysed for an inclined plane channel with thermal boundary conditions such that the requirement $\mathbf{g}' \times \boldsymbol{\nabla}'T = 0$ is not satisfied. As stated by the above necessary condition, this case is not compatible with a parallel flow solution.

4. Nonparallel flow in an inclined plane channel

Let us consider an infinitely-wide inclined plane channel such that $\mathbf{g}' = (g_x, 0)$. A drawing of the channel is given in Fig. 2. Let us consider the following thermal boundary conditions: the duct walls $y = 0$ and $y = b$ have uniform temperatures T_1 and T_2, respectively. Obviously, these boundary conditions cannot yield an identically vanishing heat flux in the y-direction. As a consequence, $\mathbf{g}' \times \boldsymbol{\nabla}'T$ cannot vanish at every position

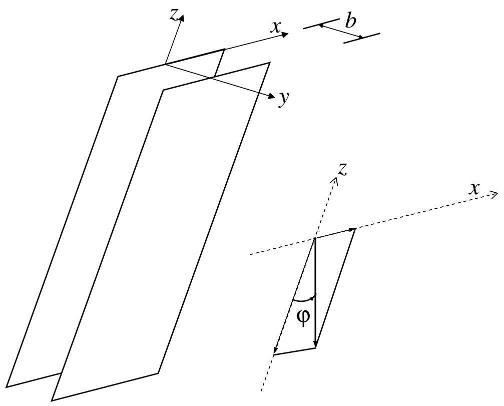

Fig. 2. Drawing of the system and of the coordinate axes for the nonparallel flow solution.

in the channel section. Then, on account of the necessary condition proved in Section 2, one can conclude that parallel flow cannot occur in this case.

As is shown in Fig. 2, the tilt angle between the z-axis and the gravitational field is

$$\varphi = \arcsin\left(\frac{g_x}{\sqrt{g_x^2 + g_z^2}}\right), \tag{21}$$

and is such that $-\pi/2 \leqslant \varphi \leqslant \pi/2$, while the component g_z of the gravitational acceleration is negative. Let us assume that the viscous dissipation term can be neglected in the energy balance equation. Since the plane channel has infinite width in the x-direction, it is conceivable to consider the fields $\mathbf{U}$, T and P as invariant in the x-direction. As a consequence, the derivatives $\partial U_x/\partial x$, $\partial U_z/\partial x$, $\partial T/\partial x$ and $\partial A/\partial x$ vanish. Therefore, the mass balance Eq. (7) yields

$$\frac{\partial U_y}{\partial y} = 0. \tag{22}$$

Eq. (22) allows one to conclude that U_y cannot depend on y and, since the fluid velocity vanishes at the walls $y = 0$ and $y = b$, that the component U_y must vanish everywhere. Therefore, the governing equations (10), (14) and (15) can be simplified, namely

$$\Delta\varrho(T)g_x + \mu\frac{d^2U_x}{dy^2} = 0, \tag{23}$$

$$\frac{dA}{dy} = 0, \tag{24}$$

$$\Delta\varrho(T)g_z - B + \mu\frac{d^2U_z}{dy^2} = 0, \tag{25}$$

$$\frac{d^2T}{dy^2} = 0. \tag{26}$$

Eq. (24) implies that A is merely a constant. Eq. (26) allows one to infer that the temperature distribution is given by

$$T(y) = T_1 + \frac{T_2 - T_1}{b}y. \tag{27}$$

As it has been shown in Ref. [14], the most reliable choice of the reference temperature T_0 is the one which ensures that the reference mass density $\varrho_0 = \varrho(T_0)$ is the mean mass density in a duct section. If the equation of state for the fluid is such that $\varrho(T)$ is assumed to be a linear function, then it can be easily checked that T_0 coincides with the mean temperature in a duct section. However, by considering Eq. (18), one can conclude that the assumption of a linear equation of state requires that β must be treated as a constant, so that

$$\varrho(T) = \varrho_0 \exp[-\beta(T - T_0)]. \tag{28}$$

Moreover, at each position, the difference $T - T_0$ is usually assumed to be so small that the Taylor expansion of the right hand side of Eq. (28) around $T = T_0$ can be truncated to the first order in $T - T_0$. In the following, this second assumption will not be invoked, so that the equation of state for the fluid is given by Eq. (28). Therefore, the reference temperature T_0 can be evaluated by employing the constraint

$$0 = \int_0^b [\varrho(T(y)) - \varrho_0]dy = \varrho_0 \int_0^b \{\exp[-\beta(T(y) - T_0)] - 1\}dy. \tag{29}$$

By substituting Eq. (27) in Eq. (29), one obtains

$$T_0 = T_1 + T_2 + \frac{1}{\beta}\ln\left[\frac{\beta(T_2 - T_1)}{\exp(\beta T_2) - \exp(\beta T_1)}\right]. \tag{30}$$

Let us define the dimensionless variables

$$v = \frac{U_x}{U_0}, \quad u = \frac{U_z}{U_0}, \quad Y = \frac{y}{b}, \quad \Lambda = -\frac{b^2B}{\mu U_0},$$
$$\Xi = \beta(T_2 - T_1), \quad Gr = \frac{\varrho_0^2 D^3 g_z \Xi}{\mu^2}, \quad Re = \frac{\varrho_0 U_0 D}{\mu},$$
$$\Gamma = \frac{Gr}{Re} = \frac{\varrho_0 D^2 g_z \Xi}{\mu U_0}, \tag{31}$$

and the dimensionless function

$$F(Y) = \frac{1}{4}\left[\frac{\exp(-\Xi Y)}{1 - \exp(-\Xi)} - \frac{1}{\Xi}\right]. \tag{32}$$

Function $v(Y)$ represents the secondary dimensionless velocity. In Eq. (31), U_0 denotes the mean velocity in a channel section, namely

$$U_0 = \frac{1}{b}\int_0^b U_z dy. \tag{33}$$

By employing Eq. (31), Eqs. (23) and (25) can be rewritten as follows:

$$\frac{d^2v}{dY^2} = \Gamma F(Y)\tan\varphi, \tag{34}$$

$$\frac{d^2u}{dY^2} = -\Gamma F(Y) - \Lambda, \tag{35}$$

The solution of Eqs. (34) and (35) is uniquely determined by taking into account the no-slip conditions

$$u(0) = u(1) = v(0) = v(1) = 0 \tag{36}$$

and the additional constraint

$$\int_0^1 u(Y)dY = 1, \tag{37}$$

which is induced by the definition of mean velocity U_0 in a channel cross section. This solution can be expressed as

$$v(Y) = -\frac{\Gamma\tan\varphi}{8\Xi^2[\exp(\Xi)-1]}\{Y(2+\Xi-\Xi Y) + \exp(\Xi)(1-Y)(2-\Xi Y) - 2\exp[\Xi(1-Y)]\}, \tag{38}$$

$$u(Y) = \frac{\Lambda}{2}Y(1-Y) + \frac{\Gamma}{8\Xi^2[\exp(\Xi)-1]}\{Y(2+\Xi-\Xi Y) + \exp(\Xi)(1-Y)(2-\Xi Y) - 2\exp[\Xi(1-Y)]\}, \tag{39}$$

$$\Lambda = 12 + \frac{\Gamma}{4\Xi^3}\left[12 + \Xi^2 - 6\Xi\coth(\Xi/2)\right]. \tag{40}$$

Eqs. (38)–(40) allow one to conclude that the nonparallel flow considered in this section can be described as the superposition of a parallel streamwise flow given by $u(Y)$ with a secondary flow given by $v(Y)$.

The occurrence of flow reversal phenomena can be revealed by an analysis of the streamwise flow component, i.e. by inspecting the dimensionless function $u(Y)$. It is easily verified that, if $\Gamma > 0$, flow reversal occurs next to the wall $Y = 1$. On the other hand, if $\Gamma < 0$, flow reversal occurs next to the wall $Y = 0$. By assuming $T_2 > T_1$, i.e. $\Xi > 0$, the case $\Gamma > 0$ corresponds to downward flow ($U_0 < 0$) and the case $\Gamma < 0$ corresponds to upward flow ($U_0 > 0$). Obviously, the reverse holds if one assumes $T_1 > T_2$, i.e. $\Xi < 0$. The onset of flow reversal for $\Gamma > 0$ occurs next to $Y = 1$ when, at this boundary, the sign of du/dY changes from negative to positive. The threshold value of Γ for the onset of flow reversal at $Y = 1$ is easily evaluated by employing Eqs. (39) and (40), namely

$$\Gamma_{fr} = \frac{24\Xi^3[\exp(\Xi)-1]}{6 + 2\exp(\Xi)(\Xi-3) + \Xi(\Xi+4)}. \tag{41}$$

For $\Gamma < 0$, flow reversal next to $Y = 0$ arises when, at this boundary, the sign of du/dY changes from positive to negative. Then, on account of Eqs. (39) and (40), the threshold value of Γ for the onset of flow reversal at $Y = 0$ is given by

$$\widetilde{\Gamma}_{fr} = -\frac{24\Xi^3[\exp(\Xi)-1]}{\exp(\Xi)(\Xi^2 - 4\Xi + 6) - 2(\Xi+3)}. \tag{42}$$

To summarize, flow reversal occurs next to the boundary $Y = 1$ for positive values of Γ such that $\Gamma > \Gamma_{fr}$; flow reversal occurs next to the boundary $Y = 0$ for negative values of Γ such that $\Gamma < \widetilde{\Gamma}_{fr}$. A representation of the states of flow reversal in the (Γ, Ξ)-plane is given in Fig. 3. This figure reveals that a symmetry in the onset of flow reversal for upward flow and for downward flow exists for $\Xi \to 0$. In fact, in this limit $|\widetilde{\Gamma}_{fr}| = \Gamma_{fr} = 288$. On the other hand, a slight departure from this symmetry takes place for any nonvanishing value of Ξ. Fig. 3 shows that $|\widetilde{\Gamma}_{fr}| < \Gamma_{fr}$ for $\Xi > 0$ and $|\widetilde{\Gamma}_{fr}| > \Gamma_{fr}$ for $\Xi < 0$, i.e. flow reversal can arise for upward flow with values of $|\Gamma|$ slightly smaller than those necessary for the occurrence of flow reversal in the case of downward flow.

In the (x,z)-plane, the local bending of the velocity vector with respect to the z-direction can be represented by the angle $\psi(Y)$ such that

$$\cos[\psi(Y)] = \frac{U_z}{(U_x^2 + U_z^2)^{1/2}} = \pm\frac{u(Y)}{[u(Y)^2 + v(Y)^2]^{1/2}},$$
$$\sin[\psi(Y)] = \frac{U_x}{(U_x^2 + U_z^2)^{1/2}} = \pm\frac{v(Y)}{[u(Y)^2 + v(Y)^2]^{1/2}}, \tag{43}$$

where the sign "+" must be used for upward flow ($U_0 > 0$), while the sign "−" must be used for downward flow ($U_0 < 0$). The behaviour of the functions $\cos[\psi(Y)]$ and $\sin[\psi(Y)]$ is represented in Fig. 4 for the special case $\varphi = \pi/3$ and $\Xi = 0.1$, with positive values of Γ. These plots allow one to infer the following features of the velocity field in this example, which corresponds to a downward flow with $T_2 > T_1$. In the case $\Gamma = 10$, $\cos[\psi(Y)]$ is almost equal to -1 and $\sin[\psi(Y)]$ is almost equal to 0 at every position inside the channel, so that the velocity **U** is a field almost parallel to the z-axis with the opposite direction ($\psi(Y) \approx \pi$). Indeed, for $\Gamma = 10$, buoyancy has very small influence on the velocity field and secondary flow is almost negligible. In the cases $\Gamma = 500$ and $\Gamma = 1000$, $\cos[\psi(Y)]$ is positive next to the wall $Y = 1$, i.e. flow reversal occurs, since positive values of $\cos[\psi(Y)]$ correspond to negative values of $u(Y) = U_z/U_0$. For all the values of Γ, $\cos[\psi(1/2)] = -1$ and $\sin[\psi(1/2)] = 0$, i.e. **U** is antiparallel to the z-direction at the midplane $Y = 1/2$ ($\psi(1/2) = \pi$). For all the values of Γ, $\sin[\psi(Y)] > 0$ in the half-channel $0 < Y < 1/2$, while $\sin[\psi(Y)] < 0$ in the half-channel $1/2 < Y < 1$. In other

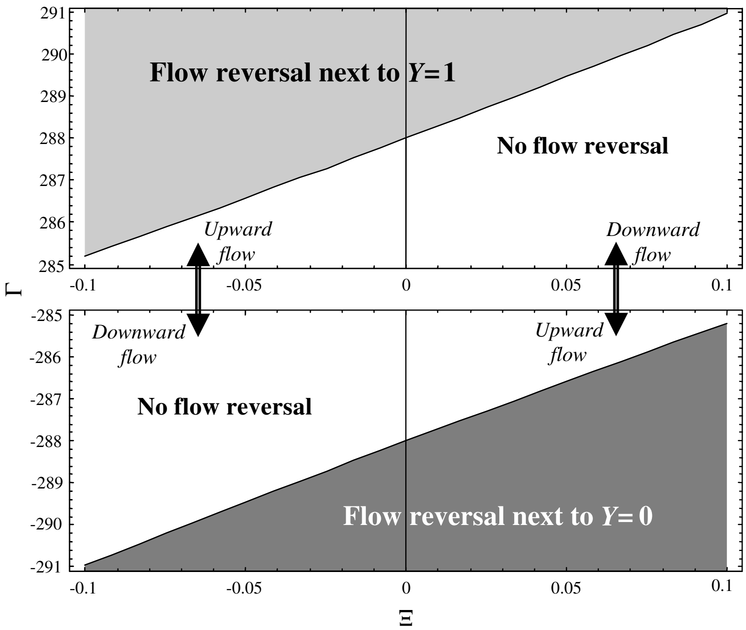

Fig. 3. Flow reversal regions in the (Γ, Ξ)-plane.

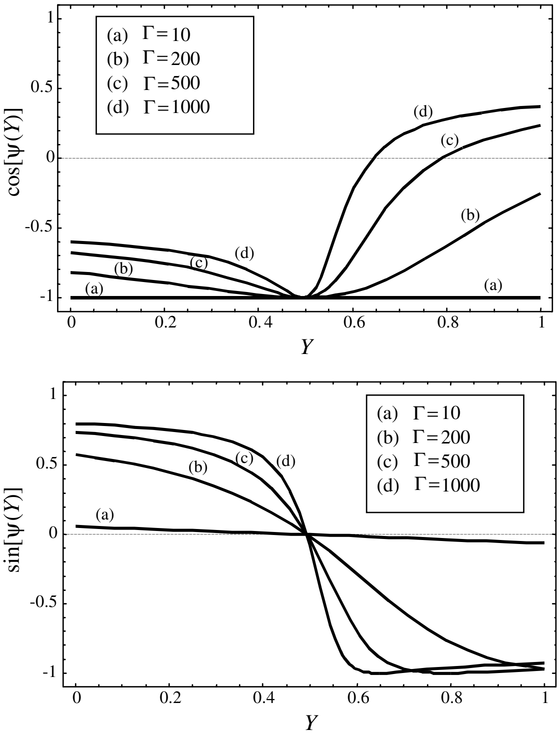

Fig. 4. Plots of $\cos[\psi(Y)]$ and $\sin[\psi(Y)]$ versus Y for $\varphi = \pi/3$ and $\Xi = 0.1$.

words, U_x is positive for $0 < Y < 1/2$ and negative for $1/2 < Y < 1$. As expected according to the direction of buoyancy forces, this circumstance corresponds to a secondary flow parallel to the x-axis in the half-channel $0 < Y < 1/2$ (i.e. next to the cold wall) and antiparallel to the x-axis in the half-channel $1/2 < Y < 1$ (i.e. next to the hot wall).

In the limit $\Xi \to 0$, Eqs. (38)–(40) reveal that the parameter Λ becomes independent of Γ, while the dimensionless velocity components still depend on Γ. In this limit, it is easily verified that the parallel flow component $u(Y)$ is compatible with the solution presented in Refs. [15] and [16], that refers to the case of vertical channel flow with a linear equation of state for the fluid.

5. Conclusions

Combined forced and free flow in an inclined duct for a fluid with constant properties has been analysed under conditions of fully developed regime, by invoking the Boussinesq approximation. A statement defining the necessary condition for the occurrence of parallel flow in an inclined duct has been proved. An example has been treated: an inclined plane channel with thermal boundary conditions incompatible with parallel flow. The solution of the momentum and energy balance equations has been found analytically, with reference to an exponential equation of state. The limit of a linear equation of state is recovered when the dimensionless parameter $\Xi = \beta(T_2 - T_1)$ tends to zero.

In the discussed example, incompatibility with parallel flow implies the existence of a secondary-flow velocity component, parallel to the channel walls and orthogonal to the streamwise direction. This secondary flow is superposed to a parallel flow in the streamwise direction. Through an analysis of the streamwise component of the velocity field, the conditions for the occurrence of flow reversal phenomenon have been obtained, both for the case of upward flow and for the case of downward flow. Moreover, the local bending angle of the velocity vector with respect to the streamwise direction has been evaluated.

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